

The quantum Neumann model: refined semiclassical results.

Marc Bellon and Michel Talon*

September 2005

Abstract

We extend the semiclassical study of the Neumann model down to the deep quantum regime. A detailed study of connection formulae at the turning points allows to get good matching with the exact results for the whole range of parameters.

LPTHE-05-29

*LPTHE, CNRS et Universités Paris VI–Paris VII (UMR 7589), Boîte 126, 4 place Jussieu,
F-75252 PARIS CEDEX 05

1 Introduction

The Neumann model is an interesting example of integrable model, both classically and quantum mechanically, which describes the motion of a point on a sphere subject to a harmonic potential. In the quantum case a lot of effort has been devoted to study the semi-classical approximation [1], since it deals with algebro-geometric objects which appear naturally in the classical theory. However this study becomes more interesting and concrete when one has precise numbers to state effectively the Bohr-Sommerfeld conditions and compare them with exact spectra. In a previous works [2, 3], we showed how to solve numerically the separated Schrödinger equation to obtain numerical values of the energy and compared this to a semiclassical computation valid for a large radius of the sphere. This case corresponds to the localization of the particle around the antipodal minima of the potential. These localized states come in pairs which are split by the tunneling probability between the two poles. The two states are either symmetric or antisymmetric under exchange of the poles.

When the sphere radius shrinks, the states become delocalized and the degeneracy is lifted by a finite quantity. In the limit of zero radius, the potential becomes irrelevant and the energy is the one of the free particle on the sphere $1/2j(j+1)$. The symmetric and antisymmetric solutions go to states with j differing by one. When flowing between the large radius and small radius cases, the matching conditions used in the WKB analysis must change, in particular to explain this degeneracy lifting. We study in this work the evolution of the matching phase which either goes from $\pi/4$ to 0 through $\pi/8$ for a symmetric wave function or from $\pi/4$ to $\pi/2$ through $3\pi/8$ for an antisymmetric wave function.

In a first part, we study the zero radius limiting case, where the energy is reproduced by the WKB method up to a small constant. Then we detail the numerical evolution of the phase. In a last part, we explain this evolution through the asymptotic analysis of a simplified model.

2 Semiclassical analysis of spheroidal harmonics.

Even if the Neumann model has been generalized to arbitrary dimensions, there are no really new phenomena appearing above the three dimensions to which we limit ourselves in this work. The parameters of the Neumann model are the oscillator strengths which can be reduced without loss of generality to be 0, 1 and y . The Schrödinger equation separates using the Neumann coordinates, yielding the one-dimensional equation:

$$\left(\frac{d^2}{dt^2} + \frac{1}{2} \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-y} \right) \frac{d}{dt} - \left(\frac{v - f_2 - f_3}{t} + \frac{f_2}{t-1} + \frac{f_3}{t-y} \right) \right) \Psi(t) = 0 \quad (1)$$

In this equation, f_2 and f_3 are the eigenvalues of the conserved quantities which have to be determined and v is $r^2/(4\hbar^2)$ where r is the radius of the sphere. The energy is $E = 2\hbar^2(f_2 + yf_3)$. In the sequel, we take $\hbar = 1$. The points 0, 1 and y are regular singularities of eq. (1) with exponents 0 and $1/2$, corresponding to solutions with monodromies ± 1 . In order to recover a well defined solution on the sphere with definite parity properties under the three possible reflexions, the solution must have definite monodromies at the three singularities [2]. An even solution is thus a solution with monodromy 1 at 0, 1 and y , that is a function which is analytic on the whole complex plane. If we want an odd solution under $x_1 \rightarrow -x_1$, we factor \sqrt{t} in Ψ and search for an analytic solution to the equation:

$$\left(\frac{d^2}{dt^2} + \frac{1}{2} \left(\frac{3}{t} + \frac{1}{t-1} + \frac{1}{t-y} \right) \frac{d}{dt} - \left(\frac{v - f_2 - f_3 + 1/4 + 1/(4y)}{t} + \frac{f_2 - 1/4}{t-1} + \frac{f_3 - 1/(4y)}{t-y} \right) \right) \Psi(t) = 0 \quad (2)$$

Similar equations can be written for the six other possible combinations of parities.

In our previous work [3], we showed how the semiclassical analysis nicely fits the exact numerical spectra of the Neumann model in the large v limit, when the point is confined around the poles. To extend the analysis to the whole range of values for v , we first consider the $v = 0$ case, corresponding to the spheroidal harmonics. The energy is known exactly in this case, since it is simply the energy of a free particle on the sphere, $\frac{1}{2}j(j+1)$.

In this case, the potential simplifies since the numerator is of degree one:

$$V = -\frac{E}{2} \frac{t-b}{t(t-1)(t-y)} \quad (3)$$

The Bohr–Sommerfeld integrals are thus elliptic integrals, with the four special points 0, 1, b and y corresponding to the vertices of the period rectangle, and a pole at infinity. We have two cases to study according to the position of b with respect to 1.

The quantification conditions, in the case with all monodromies equal to 1, are either ones of:

$$b > 1, \quad \int_0^1 \sqrt{V} dt = m\pi, \quad \int_b^y \sqrt{V} dt = (n + \frac{1}{4})\pi \quad (4)$$

$$b < 1, \quad \int_0^b \sqrt{V} dt = (m + \frac{1}{4})\pi, \quad \int_1^y \sqrt{V} dt = n\pi \quad (5)$$

These quantification conditions involve the usual $\pi/4$ phase factor coming from the asymptotics of the Airy function around the turning point b , see e.g. [4].

Adding the two Bohr–Sommerfeld integrals, one gets a condition which does not depend on the position of b with respect to 1. In fact, this sum can be

computed and does not depend on b . Indeed, the integrand is an abelian integral of the third type with simple poles at the two points over $t = \infty$. The real part of the integral on the real axis is just the sum we want to evaluate, since the integrand is purely imaginary on the complementary segments. Hence the residue theorem allows to calculate this sum. The residue is $i\sqrt{E/2}$ and since the integration contour runs through the pole, we get a contribution of $i\pi$ times the residue. The final quantification condition is therefore:

$$\pi\sqrt{E/2} = (m + n + \frac{1}{4})\pi \quad (6)$$

The semiclassical energy thus depends only on the total number of excitations and not on the individual values of m and n . The angular momentum is $j = 2(m+n)$ so that $E = 1/2(j+1/2)^2 = 1/2j(j+1)+1/8$. This differs by merely $1/8$ from the exact result and is exactly the value of L^2 which must be used in a semiclassical treatment of spherically symmetric potential according to Langer [5].

When b varies, we should be able to pass smoothly from the case of eq. (4) to the one of eq. (5). The additional phases therefore go from 0 to $\pi/4$ and from $\pi/4$ to 0, but their sum should remain constant for the semiclassical energy to be completely independent of b . In particular, in the case $b = 1$, we should have for symmetry reason the same $\pi/8$ phase on both sides. We shall show below that this is indeed the case.

3 Numerical investigations.

In order to study the variation of the phases appearing in the connection formulae at the turning points, we numerically solve the equation (1) using the methods of [2]. From the values of f_2 and f_3 , we calculate the zeros b_1 and b_2 of the potential, which can be written:

$$V = -v \frac{(t - b_1)(t - b_2)}{t(t - 1)(t - y)} \quad (7)$$

The Bohr–Sommerfeld integrals are taken on intervals where the potential is positive, with boundary points in the set $\{0, 1, y, b_1, b_2\}$. We numerically evaluate them and plot their differences with naive expectations in units of π . In the following plots, we study the case where the excitation numbers are $n = 0$ in the interval $[0, 1]$ and $m = 4$ in the interval $[1, y = 2]$. We obtain similar results for other values of the parameters n , m and y . The plot (Figure 1) clearly shows that the additional phase in the interval $[1, 2]$ goes from 0 when b_2 is smaller than 1 to a value approaching $\pi/4$ in the other case, with the variation stronger when b_2 crosses 1. To isolate the dependence on b_1 , we plot the sum of the two integrals which should not depend on the position of b_2 with respect to 1. It varies from $\pi/4$ for negative b_1 to $\pi/2$ for positive b_1 .

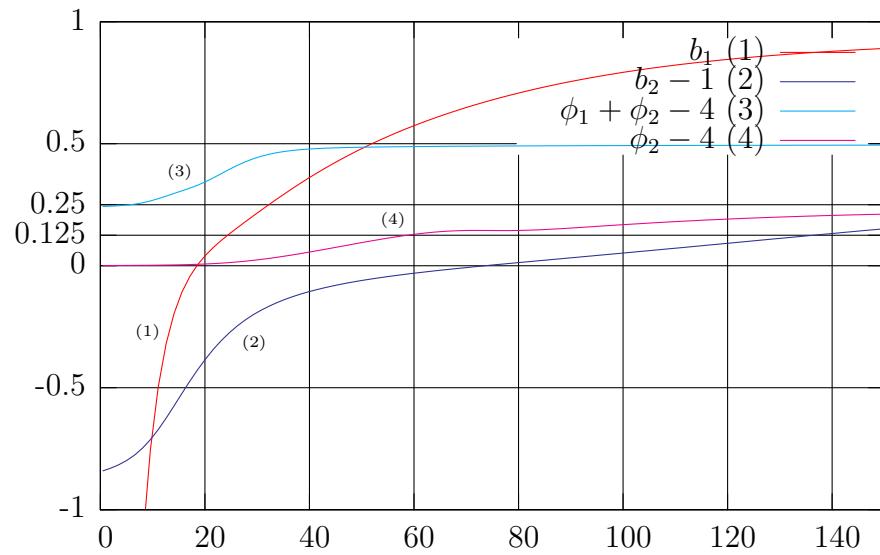


Figure 1: Variation of b_1 , b_2 and phases as function of v .

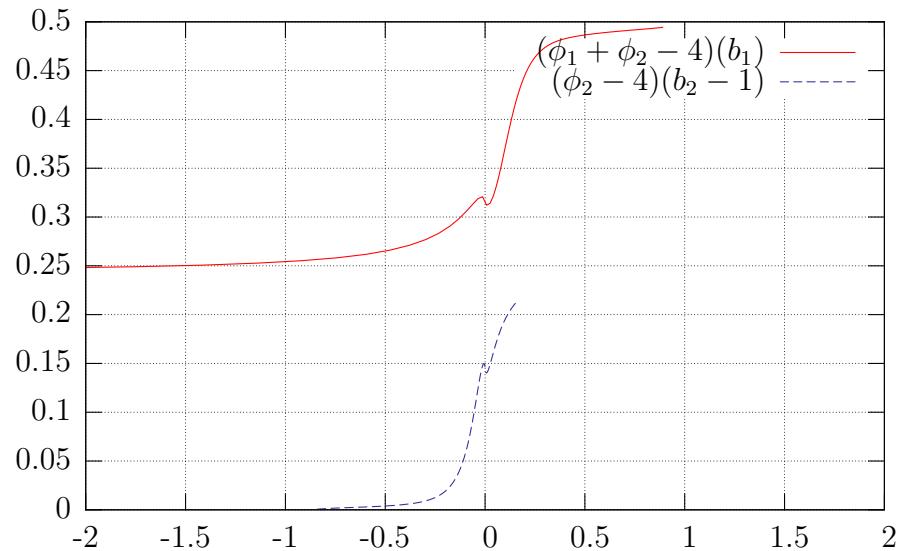


Figure 2: Phases as a function of the roots b_i .

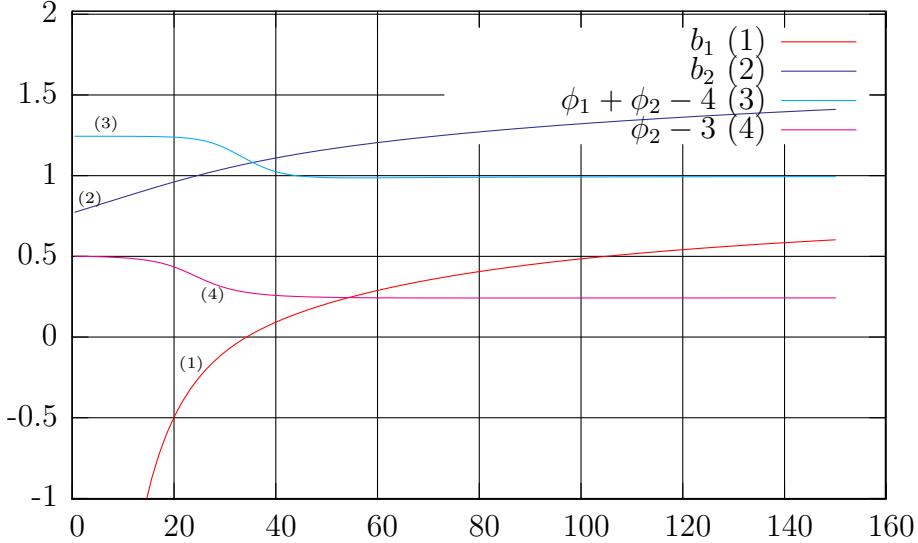


Figure 3: Variation of b_1 , b_2 and phases as function of v .

In order to show the details of the transition region, we next plot (Figure 2) the relevant phases as functions of b_1 or $b_2 - 1$. We see that the dependence of the phase is not monotonic, but there is a small "glitch" just at $b_1 = 0$ or $b_2 = 1$. We shall see below that this can indeed be explained by analytical studies. We also see that the value of the phase is not precisely $\pi/8$ at this point of transition.

We further study a case with different monodromies and values $n = 1$ and $m = 3$ for the number of zeros in each of the subintervals. Precisely in our example we choose monodromy -1 at 0 and 1 and monodromy $+1$ at $y=2$, and obtain the plot (Figure 3). In this case, the additional phases decrease from $\pi/2$ to $\pi/4$ through $3\pi/8$ in the transition region. This difference of behavior with the case of monodromies $+1$ perfectly explains the lifting of degeneracy which occurs at small v between the two degenerate levels at large v . The relative plot (Figure 4) shows that in this case the transition is smoother, which will be explained in the next section.

4 Theoretical study of the transition region.

To understand the behavior of the phases in the transition region we consider for example the region around the singularity $t=0$ and formulate a simplified version of eq.(1) for the case of monodromy 1 or of eq.(2) for the case of monodromy -1 . The simplified model uses that the terms having poles at $t=1$ or $t=y$ will not vary much as long as t remains close to 0, and doesn't approach the next pole 1. Hence we simply replace these polar terms by constants, and study the

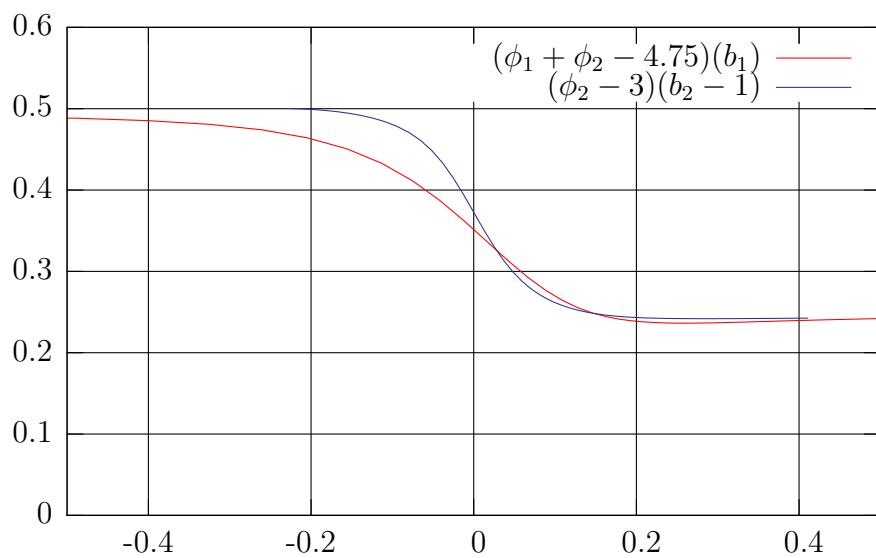


Figure 4: Phases as a function of the roots b_i .

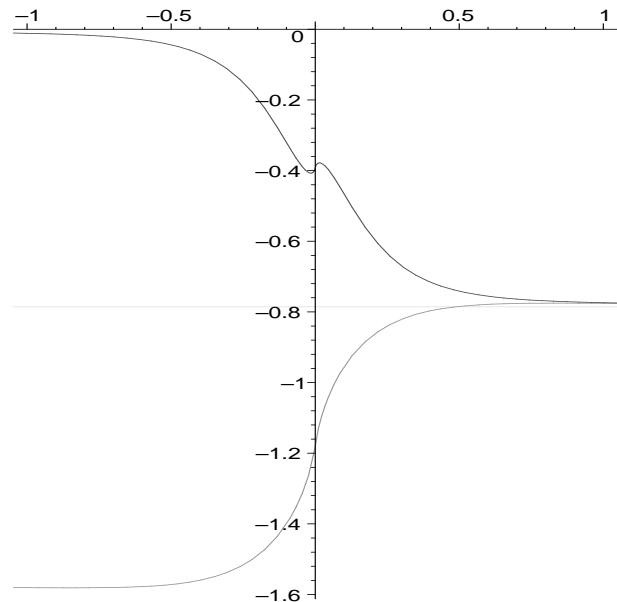


Figure 5: Variation of the phases with respect to F .

equations, where $f_1 = v - f_2 - f_3$:

$$\left(\frac{d^2}{dt^2} + \left(\frac{1}{2t} + A \right) \frac{d}{dt} - \left(\frac{f_1}{t} + B \right) \right) \Psi(t) = 0 \quad (8)$$

$$\left(\frac{d^2}{dt^2} + \left(\frac{3}{2t} + A \right) \frac{d}{dt} - \left(\frac{f_1 - A/2}{t} + B \right) \right) \Psi(t) = 0 \quad (9)$$

Equations of this form can be readily related to the confluent hypergeometric differential equation, since there is clearly a regular singularity at $t = 0$ and an irregular one at $t = \infty$.

Recall the differential equation satisfied by the confluent hypergeometric functions [7]:

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0 \quad (10)$$

The solution analytic at the regular singularity $z = 0$ is:

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{a(a-1)z^2}{2!b(b-1)} + \dots$$

One can find its asymptotics at the irregular singularity $z = \infty$ by relating it to the Whittaker functions which have simple asymptotics there, through the Mellin-Barnes transformation [6]. The result is given [7] by;

$$\frac{M(a, b, z)}{\Gamma(b)} = \frac{e^{\pm i\pi a} z^{-a}}{\Gamma(b-a)} \left(1 + O\left(\frac{1}{z}\right) \right) + \frac{e^z z^{a-b}}{\Gamma(a)} \left(1 + O\left(\frac{1}{z}\right) \right) \quad (11)$$

where the plus sign is taken in the sector $-\pi/2 < \arg z < 3\pi/2$ and the minus sign in the sector $-3\pi/2 < \arg z \leq -\pi/2$.

To get eqs.(8,9) into the form eq.(10) we take $\Psi(t) = \exp(\alpha t)w(z)$ and $t = \beta z$ and observe one gets exactly the confluent hypergeometric equation taking:

$$\alpha = -\frac{A}{2} + i\sqrt{B - \frac{A^2}{4}}, \quad \beta = \frac{i}{\sqrt{4B - A^2}}$$

Then one obtains, starting from eq. (8) that:

$$a = \frac{1}{4} - \beta(f_1 - \frac{1}{4}A), \quad b = \frac{1}{2}$$

and similarly starting from eq. (9) that:

$$a = \frac{3}{4} - \beta(f_1 - \frac{5}{4}A), \quad b = \frac{3}{2}$$

Finally remembering that we need precisely the solution which is regular at $t=0$ in both cases, the corresponding solution is thus $M(a, b, z)$, whose asymptotic expansion we know from eq.(11), and which can be directly compared with the

semi-classical approximation. Of course the validity of this procedure rests on the hypothesis that t is large enough so that one can apply the asymptotic analysis, still small enough that the next pole, $t = 1$ remains distant. This is the case when β is small, and is thus justified in our case for large B , that is when f_2 and f_3 are large enough. Note that in this case A is small compared to B . In the two cases, a takes the form $\frac{1}{4} - iF$ or $\frac{3}{4} - iF'$.

Applying the asymptotic formula (11) we get for large positive t , respectively:

$$\Psi(t) \simeq \frac{1}{t^{1/4}} \cos(\omega t - \frac{\pi}{8} + F \ln(2\omega t) - \Im \ln \Gamma(\frac{1}{4} + iF)) \quad (12)$$

$$\Psi(t) \simeq \frac{1}{t^{3/4}} \cos(\omega t - \frac{3\pi}{8} + F' \ln(2\omega t) - \Im \ln \Gamma(\frac{3}{4} + iF')) \quad (13)$$

with $\omega = \sqrt{B - A^2/4}$. We then have to compare these results with the phase we obtain from the WKB analysis of eqs. (8), which covers the cases of the two monodromies. The semiclassical action is given by:

$$\int_0^t \sqrt{\frac{Bt' + f_1}{t'}} dt' = \sqrt{t(Bt + f_1)} + \frac{f_1}{2\sqrt{B}} \ln(1 + 2Bt/f_1 + 2\sqrt{(1 + Bt/f_1)(Bt/f_1)})$$

In the limit of large Bt/f_1 , this is simply

$$\omega t + F \ln(2\omega t) + F - F \ln(F)$$

with the same quantities ω and F defined above, but taken with $A = 0$. If $A = 0$, we deduce that the phase difference between the semiclassical result and the exact one are therefore:

$$\Delta\phi_+ = \frac{\pi}{8} + \Im \ln \Gamma(\frac{1}{4} + iF) - F \ln(F) + F \quad (14)$$

$$\Delta\phi_- = \frac{3\pi}{8} + \Im \ln \Gamma(\frac{3}{4} + iF') - F' \ln(F') + F' \quad (15)$$

In fact, $F = F'$ when $A = 0$. The variation of the phases $\Delta\phi_{\pm}$ are plotted in (Figure 5), which shows that this analytical study recovers the limit of the phases found numerically in the preceding section. This plot shows a striking similarity with the above numerical results, complete with the ‘glitch’ around 0 in $\Delta\phi_+$, which comes from the singularity of the derivative of $F \ln(F)$ at $F = 0$. One could worry on the possibility to make a proper match when A is non zero. However, the difference on ω is quadratic in A and is further divided by \sqrt{B} , so that the corresponding phase difference cannot become significant over the range necessary to obtain a proper match. The term proportional to the logarithm of ωt is linear in A , but the slow variation of the logarithm makes it also benign, so that the only contribution which is significant is in the phase of the Γ function. This should explain the fact that when the b_i cross their critical line, which

corresponds to $f_1 = 0$, we obtain phases which are different from the $\pi/8$ or $3\pi/8$ which we obtain in the simpler treatment with $A = 0$. Changing the sign of t corresponds to keeping B while changing the signs of f_1 and A . Our formulae for the $\Delta\phi_{\pm}$ show that the phases on both side of a singularity add up to $\pi/4$ or $3\pi/4$ as we supposed in section 2 for the invariance of the semiclassical energy at $v = 0$

5 Conclusion.

The combined results of [3] and the present work show that it is possible to reproduce the entire spectrum of the quantum Neumann model with semiclassical methods in the whole range of parameters. The accuracy remains good even for low lying levels, but finite differences remain between the semiclassical and exact results. The new feature in the realm of semiclassical studies is that the known facts about the asymptotic behaviour of confluent hypergeometric functions allows to properly model the transition region between different types of boundary conditions. This introduces contributions which cannot be reduced to a discrete Maslov index.

References

- [1] D. Gurarie, *Quantized Neumann problem, separable potentials on S^n and the Lamé equation.* J. Math. Phys. 36 (1995), pp. 5355–5391.
- [2] M. Bellon and M. Talon, Spectrum of the quantum Neumann model. Phys. Lett. A 337 (2005), pp. 360–368.
- [3] M. Bellon and M. Talon, The quantum Neumann model: asymptotic analysis. hep-th/0507207.
- [4] W. Pauli, *Pauli lectures on physics 5 Wave mechanics.* MIT Press, 1973.
- [5] Rudolph E. Langer, *On the Connection Formulas and the Solutions of the Wave Equation.* Phys. Rev. **51**(1938), pp. 669–676.
- [6] E.T. Whittaker and G.N. Watson. *A course of modern analysis.* Cambridge University Press, (1902).
- [7] M. Abramowitz and I. Stegun *Handbook of mathematical functions.* National Bureau of Standards 1964.